APPLICABILITY OF BOUSSINESQ APPROXIMETION TO THERMAL INSTABILITY IN A SHEAR FLOW

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APPLICABILITY OF BOUSSINESQ APPROXIMATION TO THERMAL INSTABILITY IN A SHEAR FLOW

By

Tomio ASAI and Isao NAKASUJI

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Abstract

The validity of the Boussinesq approximation is investigated by making use of numerical solutions of perturbation equations for an unstably stratified shear flow. Examinations of stability properties indicate that the Boussinesq approximation is applicable to perturbations in an atmospheric layer for which the Mach number, M, the relative range of the potential temperature, ε, and the ratio of the depth of the layer to that of the isentropic atmosphere, δ, are much smaller than unity. When the Richardson number is large, the Mach number is necessarily small under the condition of small values of ε and δ.

1. Introduction

In studies of thermal convection and other stratified fluid motions the Boussinesq approximation (Boussinesq [1903]) has been used. The most important aspect of the approximation is that variations in density can be ignored except the variability of the density in the term of external force in the equation of motion. It was proved that the Boussinesq approximation is useful in dealing with thermal convection induced by buoyancy (e. g., Rayleigh [1916]). In more recent times the conditions for the validity of the Boussinesq approximation to compressible fluids were examined by Batchelor [1953], Spiegel and Veronis [1960], Mihaljan [1962], Ogura and Phillips [1962], Malkus [1964], Gough [1969] and others. Among others Spiegel and Veronis justified the Boussinesq approximation for a compressible fluid when the depth of the fluid layer is much less compared with any scale height and the fluctuations in density and pressure induced by the motion is much smaller than the total static variations of these quantities. Based on a scale analysis, Ogura and Phillips derived the Boussinesq system of equations governing thermal convection in the atmosphere assuming that the relative range of potential temperature is small and the depth of the layer is smaller than the scale height of the adiabatic atmosphere. Extending the analysis to deal with more general cases, Gough obtained a result similar to that of Ogura and Phillips.

The objective of this paper is to investigate under what condition the Boussinesq approximation is valid for perturbation equations for an unstably stratified shear flow.

2. Basic equations

Consider an unstably stratified parallel flow in a channel whose depth and width are denoted by h and d, respectively. We use a Cartesian coordinate system, in which the x axis is parallel to the basic flow, the y axis is perpendicular to the x axis and the z axis is vertical. Assuming constant shears in the y direction as well as in the z direction and a constant lapse rate of the potential temperature, we can express the velocity, $\bar{u}$, and the potential temperature, $\bar{\theta}$, as follows.

$$\bar{u}=\Lambda_y y+\Lambda_z z, \tag{2-1}$$

$$\bar{\theta}=\theta_0-\beta\left(z-\frac{h}{2}\right), \tag{2-2}$$

where Λ_y is the horizontal shear of the flow, Λ_z is the vertical shear of the flow, β is the lapse rate of the potential temperature and θ_0 is the potential temperature at $z=h/2$, which is assumed to be constant. The pressure under the hydrostatic equilibrium is represented by

$$\bar{\pi}=\pi_0+\frac{g}{c_p\beta}\ln\left\{1-\frac{\beta}{\theta_0}\left(z-\frac{h}{2}\right)\right\}, \tag{2-3}$$

where $\bar{\pi}=(\bar{p}/p_0)^{(\gamma-1)/\gamma}$, $\bar{p}$ is the pressure under the hydrostatic equilibrium, p_0 is the pressure at $z=h/2$, $\gamma=c_p/c_v$, c_p is the specific heat at constant pressure, c_v is the specific heat at constant volume, and g is the gravitational acceleration.

The perturbation equations for small disturbances superimposed on the shear flow are written as follows.

$$\frac{\partial u'}{\partial t}+\bar{u}\frac{\partial u'}{\partial x}+v'\frac{\partial \bar{u}}{\partial y}+w'\frac{\partial \bar{u}}{\partial z}=-c_p\bar{\theta}\frac{\partial \pi'}{\partial x}+\frac{\mu}{\bar{\rho}}$$
$$\times\left\{\nabla^2 u'+\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)\right\}, \tag{2-4}$$

$$\frac{\partial v'}{\partial t}+\bar{u}\frac{\partial v'}{\partial x}=-c_p\bar{\theta}\frac{\partial \pi'}{\partial y}+\frac{\mu}{\bar{\rho}}$$
$$\times\left\{\nabla^2 v'+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)\right\}, \tag{2-5}$$

$$\frac{\partial w'}{\partial t}+\bar{u}\frac{\partial w'}{\partial x}=-c_p\bar{\theta}\frac{\partial \pi'}{\partial z}+\frac{\theta'}{\bar{\theta}}g+\frac{\mu}{\bar{\rho}}$$
$$\times\left\{\nabla^2 w'+\frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)\right\}, \tag{2-6}$$

$$\frac{1}{(\gamma-1)\bar{\pi}}\left(\frac{\partial \pi'}{\partial t}+\bar{u}\frac{\partial \pi'}{\partial x}\right)-\frac{1}{\bar{\theta}}\left(\frac{\partial \theta'}{\partial t}+\bar{u}\frac{\partial \theta'}{\partial x}\right)+w'$$
$$\times\left(\frac{1}{(\gamma-1)\bar{\pi}}\frac{\partial \bar{\pi}}{\partial z}-\frac{1}{\bar{\theta}}\frac{\partial \bar{\theta}}{\partial z}\right)=-\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right), \tag{2-7}$$

$$\frac{\partial \theta'}{\partial t}+\bar{u}\frac{\partial \theta'}{\partial x}+w'\frac{\partial \bar{\theta}}{\partial z}=\frac{\kappa\bar{\theta}}{c_p\bar{\rho}\bar{T}}\nabla^2\theta', \tag{2-8}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, u', v', w', π' and θ' are the perturbations of the velocity components in the x, y, z directions, the pressure and the potential temperature, respectively. Both the coefficient of viscosity, μ, and the coefficient of heat conduction, κ, are assumed respectively to be constants. With the use of the equation of state and the definition of π the density, $\bar{\rho}$, and the temperature, $\bar{T}$, are expressed as

$$\bar{\rho}=p_0\bar{\pi}^{\frac{1}{\gamma-1}}/R\bar{\theta}, \tag{2-9}$$

and

$$\bar{T}=\bar{\theta}\bar{\pi}, \tag{2-10}$$

where R is the gas constant.

3. Boundary conditions

We assume that both the upper and the lower boundaries as well as the lateral boundaries are fixed and smooth. Consequently the normal components of the velocity and the tangential stresses must vanish at the boundaries, i. e.,

$$w'=\frac{\partial u'}{\partial z}=\frac{\partial v'}{\partial z}=0 \quad \text{at } z=0 \text{ and } h, \tag{3-1}$$

$$v'=\frac{\partial u'}{\partial y}=\frac{\partial w'}{\partial y}=0 \quad \text{at } y=0 \text{ and } d. \tag{3-2}$$

In addition it is assumed that the upper and the lower boundaries are maintained at constant potential temperature while no heat conduction occurs at the lateral boundaries, so that

$$\theta'=0 \quad \text{at } z=0 \text{ and } h, \tag{3-3}$$

$$\frac{\partial \theta'}{\partial y}=0 \quad \text{at } y=0 \text{ and } d. \tag{3-4}$$

4. Dimensionless equations

It is convenient to express the preceding equations in the dimensionless form in terms of the characteristic values of the length, h, the velocity Δu and the potential temperature $\Delta\theta$. Here Δu is the velocity difference between the basic flows at the top and the bottom of the fluid, and $\Delta\theta$ is the difference between the potential temperatures at the top and the bottom. Then asterisked dimensionless quantities are defined as

$$\left.\begin{aligned}&x=x^*h,\quad y=y^*h,\quad z=z^*h,\quad t=t^*h/\Delta u,\\&u'=u^*\Delta u,\ v'=v^*\Delta u,\ w'=w^*\Delta u,\ \bar{u}=\bar{u}^*\Delta u,\\&\theta'=\theta^*\Delta\theta,\quad \pi'=\pi^*(\Delta u)^2/c_p\theta_0.\end{aligned}\right\}\tag{4-1}$$

Furthermore θ_0, T_0, ρ_0 and π_0 are used for nondimensionalizing the respective quantities of the basic field so that

$$\left.\begin{aligned}&\bar{\theta}=\bar{\theta}^*\theta_0,\quad \bar{T}=\bar{T}^*T_0,\\&\bar{\rho}=\bar{\rho}^*\rho_0,\quad \bar{\pi}=\bar{\pi}^*\pi_0,\end{aligned}\right\}\tag{4-2}$$

where ρ_0 and T_0 are the density and the temperature of the basic field under hydrostatic equilibrium at $z=h/2$, respectively.

Making use of (4-1) and (4-2), we can rewrite (2-4) to (2-8) as the following dimensionless equations.

$$\begin{aligned}\frac{\partial u^*}{\partial t^*}+\bar{u}^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial \bar{u}^*}{\partial y^*}+w^*\frac{\partial \bar{u}^*}{\partial z^*}=-\bar{\theta}^*\frac{\partial \pi^*}{\partial x^*}+\frac{1}{\bar{\rho}^*R_e}\\ \times\left\{\nabla^{*2}u^*+\frac{1}{3}\frac{\partial}{\partial x^*}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}+\frac{\partial w^*}{\partial z^*}\right)\right\},\end{aligned}\tag{4-3}$$

$$\begin{aligned}\frac{\partial v^*}{\partial t^*}+\bar{u}^*\frac{\partial v^*}{\partial x^*}=-\bar{\theta}^*\frac{\partial \pi^*}{\partial y^*}+\frac{1}{\bar{\rho}^*R_e}\\ \times\left\{\nabla^{*2}v^*+\frac{1}{3}\frac{\partial}{\partial y^*}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}+\frac{\partial w^*}{\partial z^*}\right)\right\},\end{aligned}\tag{4-4}$$

$$\begin{aligned}\frac{\partial w^*}{\partial t^*}+\bar{u}^*\frac{\partial w^*}{\partial x^*}=-\bar{\theta}^*\frac{\partial \pi^*}{\partial z^*}+R_i\frac{\theta^*}{\bar{\theta}^*}+\frac{1}{\bar{\rho}^*R_e}\\ \times\left\{\nabla^{*2}w^*+\frac{1}{3}\frac{\partial}{\partial z^*}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}+\frac{\partial w^*}{\partial z^*}\right)\right\},\end{aligned}\tag{4-5}$$

$$\begin{aligned}\frac{M^2}{\bar{\pi}^*}\left(\frac{\partial \pi^*}{\partial t^*}+\bar{u}^*\frac{\partial \pi^*}{\partial x^*}\right)-\frac{\varepsilon}{\bar{\theta}^*}\left(\frac{\partial \theta^*}{\partial t^*}+\bar{u}^*\frac{\partial \theta^*}{\partial x^*}\right)-w^*\\ \times\left(\frac{\delta}{(\gamma-1)\bar{\pi}^*\bar{\theta}^*}-\frac{\varepsilon}{\bar{\theta}^*}\right)=-\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}+\frac{\partial w^*}{\partial z^*}\right),\end{aligned}\tag{4-6}$$

$$\frac{\partial \theta^*}{\partial t^*}+\bar{u}^*\frac{\partial \theta^*}{\partial x^*}-w^*=\frac{\bar{\theta}^*}{P_rR_e\bar{\rho}^*\bar{T}^*}\nabla^{*2}\theta^*,\tag{4-7}$$

where R_i is the Richardson number, R_e is the Reynolds number, P_r is the Prandtl number, M is the Mach number, ε is the relative range of the potential temperature and δ is the ratio of the depth of the layer concerned to that of the isentropic atmosphere, as defined by

$$\left.\begin{aligned}&R_i=\frac{gh\Delta\theta}{\theta_0(\Delta u)^2},\quad R_e=\frac{\rho_0h\Delta u}{\mu},\\&P_r=\frac{\mu c_pT_0}{\kappa\theta_0},\quad M=\frac{\Delta u}{(\gamma RT_0)^{1/2}},\\&\varepsilon=\frac{\Delta\theta}{\theta_0},\qquad \delta=\frac{gh}{c_p\theta_0\pi_0}.\end{aligned}\right\}\tag{4-8}$$

Note that a relationship, $\varepsilon\delta=(\gamma-1)R_iM^2$, exists among the four dimensionless parameters. Dimensionless forms of (2-1) and (2-2) may be represented by

$$\bar{u}^*=\lambda y^*+z^*, \tag{4-9}$$

$$\bar{\theta}^*=1-\varepsilon\left(z^*-\frac{1}{2}\right), \tag{4-10}$$

where $\lambda=\Lambda_y/\Lambda_z$. The rest of quantities of the basic state is

$$\bar{\pi}^*=1+\frac{\delta}{\varepsilon}\ln\bar{\theta}^*, \tag{4-11}$$

$$\bar{\rho}^*=\pi^{*\frac{1}{\gamma-1}}/\bar{\theta}^*, \tag{4-12}$$

$$\bar{T}^*=\bar{\theta}^*\bar{\pi}^*. \tag{4-13}$$

It is evident that the set of equations (4-3) to (4-7) with the additional relationships (4-9) to (4-13) reduces to that under the Boussinesq approximation as the values of ε, δ and M tend to zero. In the following we will compare solutions of (4-3) to (4-7) for a wide range of the parameters of ε, δ and M with the solution of the Boussinesq system.

5. Normal mode representation

We assume perturbations of the form

$$\begin{pmatrix} u^* \\ v^* \\ w^* \\ \pi^* \\ \theta^* \end{pmatrix} = \begin{pmatrix} U^* \\ V^* \\ W^* \\ \Pi^* \\ \Theta^* \end{pmatrix} \exp\,(ik_x{}^*x^*+\sigma^*t^*) \tag{5-1}$$

where $k_x{}^*$ is the dimensionless wavenumber in the x direction, and U^*, V^*, W^*, Π^* and Θ^* are the respective dimensionless complex amplitude functions. Dimensionless frequency σ^* is generally complex and is expressed by $\sigma^*=\sigma_r{}^*+i\sigma_i{}^*$ where a positive value of $\sigma_r{}^*$ denotes the amplification rate of unstable perturbation. Substituting (5-1) into (4-3) to (4-7) and omitting asterisks for simplicity, we obtain

$$(\sigma+ik_x\bar{u})U+\frac{\partial\bar{u}}{\partial y}V+\frac{\partial\bar{u}}{\partial z}W=-ik_x\bar{\theta}\Pi$$
$$+\frac{1}{\bar{\rho}R_e}\left\{\left(-k_x{}^2+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)U+\frac{ik_x}{3}\left(ik_xU+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)\right\}, \tag{5-2}$$

$$(\sigma+ik_x\bar{u})V=-\bar{\theta}\frac{\partial\Pi}{\partial y}+\frac{1}{\bar{\rho}R_e}\left\{\left(-k_x{}^2+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)V\right.$$
$$\left.+\frac{1}{3}\frac{\partial}{\partial y}\left(ik_xU+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)\right\}, \tag{5-3}$$

$$(\sigma+ik_x\bar{u})W=-\bar{\theta}\frac{\partial \Pi}{\partial z}+\frac{R_i}{\bar{\theta}}\Theta+\frac{1}{\bar{\rho}R_e}\left\{\left(-k_x^2+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)W\right.$$
$$\left.+\frac{1}{3}\frac{\partial}{\partial z}\left(ik_xU+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)\right\}, \qquad (5\text{-}4)$$

$$(\sigma+ik_x\bar{u})\left(\frac{M^2}{\bar{\pi}}\Pi-\frac{\varepsilon}{\bar{\theta}}\Theta\right)-\left(\frac{\delta}{(\gamma-1)\bar{\pi}\bar{\theta}}-\frac{\varepsilon}{\bar{\theta}}\right)W$$
$$=-\left(ik_xU+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right), \qquad (5\text{-}5)$$

$$(\sigma+ik_x\bar{u})\Theta-W=\frac{\bar{\theta}}{P_rR_e\bar{\rho}\bar{T}}\left(-k_x^2+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Theta\,. \qquad (5\text{-}6)$$

The boundary conditions (3-1) to (3-4) are reduced to

$$W=\frac{\partial U}{\partial z}=\frac{\partial V}{\partial z}=\Theta=0 \quad \text{at } z=0 \text{ and } 1, \qquad (5\text{-}7)$$

$$V=\frac{\partial U}{\partial y}=\frac{\partial W}{\partial y}=\frac{\partial \Theta}{\partial y}=0 \quad \text{at } y=0 \text{ and } D, \qquad (5\text{-}8)$$

where $$D=d/h.$$

The differential equations (5-2) to (5-6) may be transformed to a set of algebraic equations by approximating the derivatives of U, V, W, Π and Θ with respect to y and/or z by finite differences. The numerical procedure adopted to solve the equations is essentially the same as that used by Asai [1970] and the details are described by Asai and Nakasuji [1971].

6. Results

First we obtain the amplification rate and the phase velocity as an eigenvalue of the equations (5-2) to (5-6) with use of the boundary conditions (5-7) to (5-8) for a wide range of the parameters, ε, δ and M. Then we compare the stability properties in a non-Boussinesq system with those in the Boussinesq system of which the parameters, ε, δ and M, vanish. For the sake of simplicity no horizontal shear, *i.e.* $\Lambda_y=0$, is assumed in the following discussion. We adopt $P_r=1$ and $\gamma=1.4$ for dry air.

Solid lines in Fig. 1 show variations of the amplification rate, σ_r, and of the phase velocity, $c(=-\sigma_i/k_x)$, of the unstable perturbation of $k_x=2$ and $D=\pi/2$, which is nearly of a preferred mode, with the parameter, ε, ranging from 10^{-2} to 1 for the case of $M=10^{-3}$, $R_i=10$ and $R_e=10$ in which the shear of the basic flow is small. The amplification rate and the phase velocity for the corresponding Boussinesq system are indicated by broken lines, respectively, for comparison. As is seen in Fig. 1, the amplification rate and the phase velocity in the non-Boussinesq system deviate from those in the Boussinesq

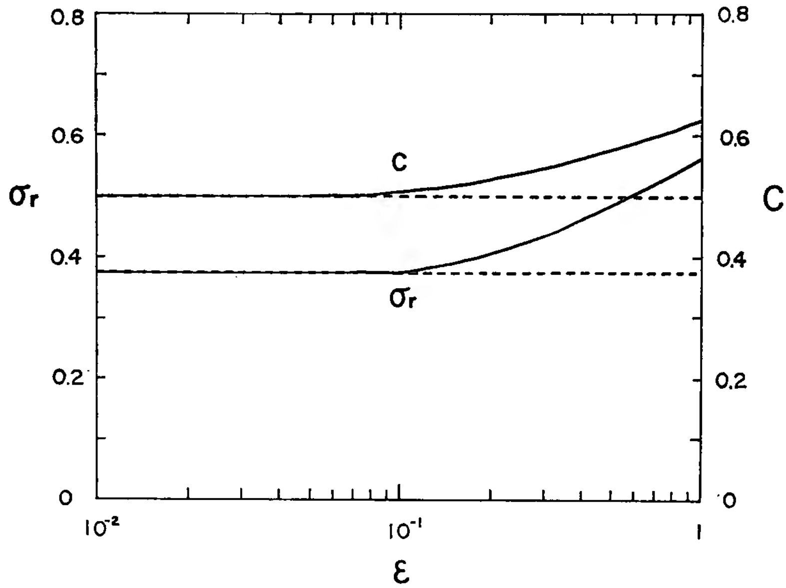

Fig. 1. Amplification rate, σ_r, and phase velocity, c, as a function of ε are indicated by solid lines, respectively, for $R_i=10$, $R_e=10$, $M=10^{-3}$, $k_x=2$ and $D=\pi/2$.
Those of the Boussinesq system are indicated by broken lines for comparison.

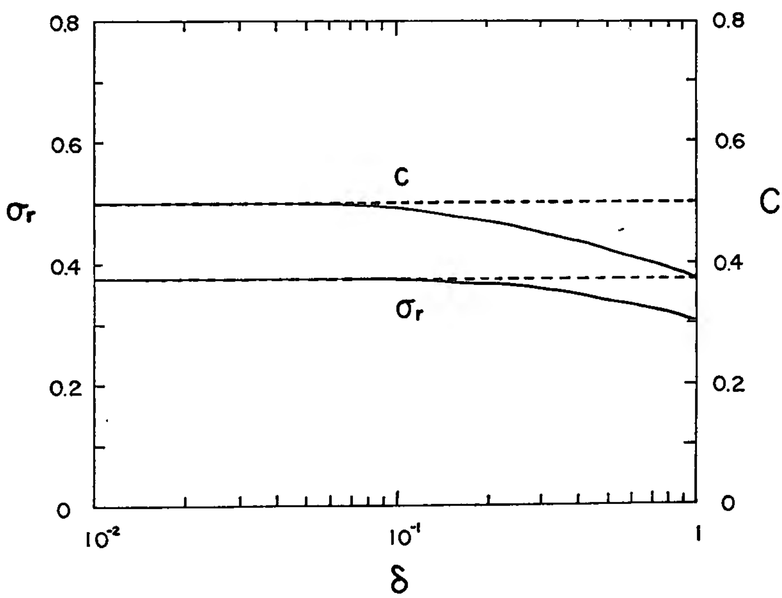

Fig. 2. Amplification rate and phase velocity as a function of δ. The others are the same as Fig. 1.

system with an increase in the value of ε. Differences between the two systems are, however, quite small for $\varepsilon<0.1$.

Variations of the amplification rate and of the phase velocity with the parameter, δ, ranging from 10^{-2} to 1 are shown in Fig. 2 for the same case as Fig. 1. As the value of δ increases both σ_r and c in the non-Boussinesq system denoted by solid lines depart from those in the Boussinesq system. However, the departure remains less than 1 per cent for $\delta<0.1$. Fig. 3 indicates varia-

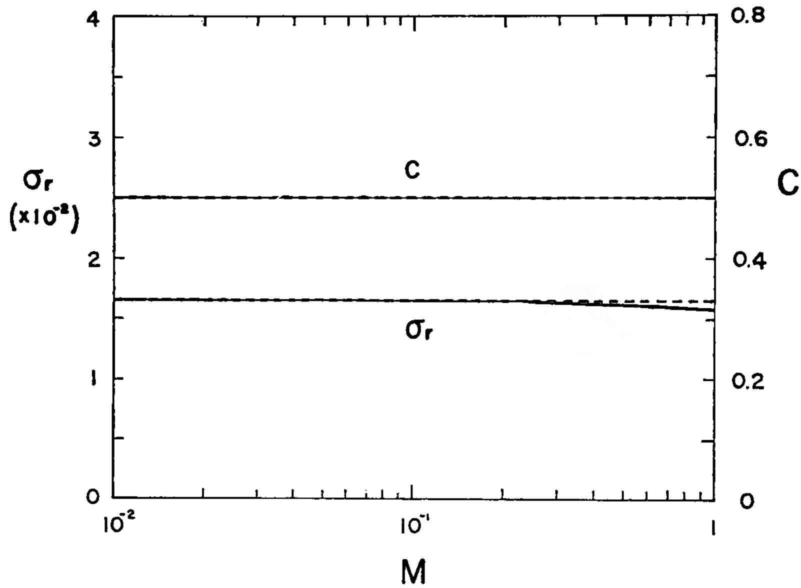

Fig. 3. Amplification rate in units of 10^{-2} and phase velocity as a function of M for $R_i=10^{-3}$, $R_e=10^5$ and $\varepsilon=\delta<10^{-2}$. The others are the same as Fig. 1.

tions of the amplification rate and of the phase velocity of the unstable perturbation of the same mode as that in Figs. 1 and 2 with the Mach number, M, for the case of $R_i=10^{-3}$ and $R_e=10^5$ in which the shear of the basic flow is large. Here the values of ε and δ are taken to be smaller than 10^{-2}. No difference appears between the phase velocities of the two systems, while the amplification rate in the non-Boussinesq system becomes a little smaller than that in the Boussinesq system as the value of M increases.

7. Conclusion

The applicability of the Boussinesq approximation to thermal instability in a shear flow is investigated by making use of a numerical solution of the perturbation equations.

In cases of the Richardson number, R_i, larger than around unity the Boussinesq approximation is accurate enough when both the relative range of the potential temperature, ε, and the ratio of the depth of the layer to that of the isentropic atmosphere, δ, are smaller than 0.1. It is noted that the Mach number, M, is necessarily small in this case. This coincides with the result obtained by Ogura and Phillips [1962] and Gough [1969] who did not take into account a shear flow. When the Richardson number is smaller than unity, a Mach number smaller than 0.1 as well as the conditions that $\varepsilon<0.1$ and $\delta<0.1$ is required to make the Boussinesq approximation applicable. On the basis of the results mentioned above, we made a study of thermal instability of a parallel flow with vertical and horizontal shears. This will be reported in a separate paper.

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